# Phantom cosmology as a simple model with dynamical complexity

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We study the Friedman-Robertson-Walker model with phantom fields modeled in terms of scalar fields. We apply the Ziglin theory of integrability and find that the flat model is nonintegrable. Then we cannot expect to determine simple analytical solutions of the Einstein equations. We demonstrate that there is only a discrete set of parameters where this model is integrable. For comparison we describe the phantoms fields in terms of the barotropic equation of state. It is shown that in contrast to the phantoms modeled as scalar fields, the dynamics is always integrable and phase portraits are contracted. In this case we find the duality relation.

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## I. INTRODUCTION

The recently available measurements of luminosity distances of the type Ia supernova (SNIa) as a function of redshift have shown that the current Universe is in an accelerating phase due to unknown form of repulsive energy [1,2]. The most popular candidate for this dark energy is the cosmological constant. On the other hand, the results of large-scale structure surveys and results of measurements of masses of galaxies give the best fit for the density parameter for matter $\Omega_{m,0}=0.3$ [3,4] (for review of cosmological parameters see Ref. [5]). Combining data from SNIa with measurements of the cosmic microwave background (CMB) radiation, we obtain $\Omega_{\Lambda,0}=0.7$ as the best fit value. The sum of the densities $\Omega_{total,0}=1.02\pm0.02$ obtained by the Wilkinson microwave anisotropy probe (WMAP) [6] agrees with the value predicted by inflation and suggests that our Universe is almost flat on the large scales. Therefore, the assumption of the flat model with the cosmological constant is in good agreement with our observations.

The acceleration of the Universe can be explained in a twofold manner. In the first approach it is postulated that there is some unknown exotic matter which violates the strong energy condition $\rho+3p\geqslant0$, where $p$ is the pressure and $\rho$ is the energy density of perfect fluid. This form of matter is called dark energy. In the past few years different scalar field models like quintessence and more recently the tachyonic scalar field have been conjectured for modeling the dark energy in terms of subnegative pressure $p>-\rho$. A scalar field with supernegative pressure $p<-\rho$ called a phantom field can formally be obtained by switching the sign of the kinetic energy in the Lagrangian for a standard scalar field. For example in the Friedmann-Robertson-Walker (FRW) model the phantom field minimally coupled to a gravity field leads to $\rho+p=-\dot{\phi}^2$, where $\rho_\phi=-1/2\dot{\phi}^2+V(\phi)$, $p_\phi=-1/2\dot{\phi}^2-V(\phi)$, and $V(\phi)$ is the phantom potential. Such a field was called the phantom field by Caldwell [7] who proposed it as a possible explanation of the observed acceleration of the current universe when $\Omega_{m,0}\gtrsim0.2$. Note that a coupling to gravity in the quintessence models was also explored [8].

The second approach called the Cardassian expansion scenario has recently been proposed by Freese and Lewis [9] as an alternative to the dark energy in order to explain the current accelerated expansion of the Universe. In this scenario the Universe is flat and matter dominated but the standard FRW dynamics is modified by the presence of an additional term $\rho^n$ such that $3H^2=\rho_{eff}=\rho+3B\rho^n$, where $H=(d\ln a)/dt$ is the Hubble parameter and $a$ is the scale factor. However, let us note that this additional term can be interpreted as a phantom field modeled by the equation of state $p=p(\rho)=[n(1+\gamma)-1]\rho$, where $\rho=\rho_{m,0}a^{-3(1+\gamma)}$. Therefore for dust matter we obtain $p=(n-1)\rho$, and $n<0$ leads to the phantom field.

The phantom scalar field can also be motivated from S-brane in string theory [10–12]. The noncanonical kinetic energy also occurs in higher-order theories of gravity and supergravity [13,14]. At first, phantom fields were introduced by bulk viscosity effects which can be present in FRW cosmology. They are equivalent to effective pressure $p_{eff}=p-3\xi H$, where $\xi$ is a bulk viscosity coefficient. It is because dissipation in general relativity is connected (in contrast to friction in classical mechanics) with the creation of the energy in the expanding Universe by the negative pressure contribution [15,16].

Without making some specific assumptions on $w(z)$ it is very difficult to constrain it from the SNIa data [17]. Because the astronomical observations do not seem to exclude the

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phantom fields which violate the weak energy condition, it is interesting to investigate the theoretical possibility to describe the dark energy in terms of a phantom field [18–35]. The Cardassian expansion with $n < 0$ which can be interpreted as the phantom fluid effect is also statistically admissible from SNIa data observations [36].

In this paper we ask what kind of dynamics can be expected from the FRW model with a phantom field. It is well known that the standard FRW model reveals some complex dynamics. The detailed studies gave us a deeper understanding of the dynamical complexity and chaos in cosmological models and resulted the in conclusion that complex behavior depends on the choice of a time parameterization or a lapse function in general relativity [37]. Castagnino *et al.* [38] showed that the dynamics of closed FRW models with a conformally coupled massive scalar field is not chaotic if considered in the cosmological time. They showed that for all initial conditions the Universe will collapse in finite time and then conclude that there is no chaos in the model. In their work the monotonously growing function is defined along a trajectory which diverge at infinity for arbitrary initial conditions. The same model was analyzed in the conformal time by Calzetta and Hasi [39] who presented the existence of chaotic behavior of trajectories in the phase space.

For the cosmological FRW model with a scalar field the kinetic energy form is indefinite, therefore, the domain admissible for motion is $R^n$. The similar situation happens in the Bianchi IX in which, as it was proved by Cushman and Sniatycki [40], trajectories have no recurrence property.

The standard methods of the chaos investigation can also be applied to the wide class of a relativistic system [41]. Motter and Letelier [42] explained that this contradiction in the results is obtained because the system under consideration is nonintegrable. Therefore we can speak about complex dynamics in terms of nonintegrability rather than deterministic chaos. The significant feature is that nonintegrability is an invariant evidence of dynamical complexity in general relativity and cosmology [43–46].

There are different motivations to study noningrability in general relativity and cosmology. One reason is the possible physical implications of the existence of complexity in the systems which for example could help to explain the formation of structures. Another reason is to develop suitable tools to study a relativistic system. The next motivation is to understand the ultimate implication of the time reparameterization.

For the FRW model with a phantom it can be shown that there is a monotonous function along its trajectories and it is not possible to obtain the Lyapunov exponents or construct the Poincaré sections. Therefore we turn to the study of nonintegrability of the phantom system and set it in a much stronger form by proving that the system does not possesses any additional and independent Hamiltonian first integrals, which are in the form of analytic or meromorphic functions. Of course, it is not the evidence of the sensitive dependence of a solution on a small change of initial conditions. However, it is the possible evidence of complexity of dynamical behavior formulated in an invariant way. We study nonintegrability in the FRW model with phantom fields and find that nonintegrability is a generic feature of this model and favors

rather nonanalytical forms of the equation of state.

It is useful to distinguish between solvability and integrability. While integrability is an intrinsic property of the system which imposes the constraints on the solutions in the phase space, the solvability is related to the existence of closed form solutions [47]. In this paper we concentrate on first integrals rather than the solutions of a system.

We study nonintegrability instead of chaos because this criterion is invariant with respect to time reparameterization. Note that while this program of a nonintegrability investigation was explicitly formulated by Motter and Letelier [42], this idea was materialized in papers by Maciejewski and Szydlowski [43–46]. Maciejewski and Szydlowski also showed that the Bianchi VIII and IX are noningrable in the sense of the nonexistence of additional analytic first integrals [48] and that the Bianchi VIII model is nonintegrable in the sense of the nonexistence of meromorphic first integrals [49]. The meromorphic function possesses only poles as its singularities; roughly speaking it is the quotient of analytic functions. The latter method is used in this paper. Ziglin proved the independently nonintegrability of the FRW closed model with a scalar field in the sense of the nonexistence of additional meromorphic first integrals [50]. In turn, Morales Ruiz and Ramis proved the nonintegrability of the Bianchi IX in the same sense [51].

For comparison we consider the FRW model with a phantom given by the barotropic equation of state which violates the weak energy condition. We obtain that this model is integrable in contrast to the previous treatment of phantom cosmology. Assuming the barotropic form of the equation of state for the phantom model we obtain the integrable dynamics at the very beginning.

## II. HAMILTONIAN DYNAMICS OF PHANTOM COSMOLOGY

We assume the model with FRW geometry, i.e., the line element has the form

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (1)$$

where

$$f(\chi) = \begin{cases} \sin\chi, & 0 \leq \chi \leq \pi & k = +1 \\ \chi, & 0 \leq \chi \leq \infty & k = 0 \\ \sinh\chi, & 0 \leq \chi \leq \infty & k = -1 \end{cases} \quad (2)$$

$k = 0, \pm 1$ is the curvature index, $0 \leq \varphi \leq 2\pi$ and $0 \leq \theta \leq \pi$ are comoving coordinates, and $\eta$ stands for the conformal time such that $dt/a \equiv d\eta$.

It is also assumed that a source of gravity is the phantom scalar field $\psi$ with a generic coupling to gravity. The gravitational dynamics is described by the standard Einstein-Hilbert action

$$S_g = -\tfrac{1}{2}m_p^2 \int d^4x \sqrt{-g}(R - 2\Lambda), \quad (3)$$

where $m_p^2 = (8\pi G)^{-1}$; for simplicity and without loss of generality we assume $4\pi G/3 = 1$. The action for the matter source is

$$S_{ph} = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ -g^{\mu\nu}\psi_\mu\psi_\nu + 2U(\psi) + \xi R\psi^2 \right]. \quad (4)$$

Let us note that the formal sign of $\|\psi\|^2$ is opposite to that which describes the standard scalar field as a source of gravity, where $U(\psi)$ is a scalar field potential. We assume

$$U(\psi) = \frac{1}{2}m^2\psi^2 + \frac{1}{4}\lambda\psi^4 \quad (5)$$

and that the conformal volume $\int d^3x$ over the spatial three-hypersurface is a unit. $\xi$ is a coupling constant of scalar field to the Ricci scalar

$$R = 6\left( \frac{\ddot{a}}{a^3} + \frac{k}{a^2} \right). \quad (6)$$

If we have the minimally coupled scalar field then $\xi=0$. We assume a nonminimal coupling of the scalar field $\xi \neq 0$.

The dynamical equation for phantom cosmology in which the phantom field is modeled by the scalar field with an opposite sign of the kinetic term in action can be obtained from the variational principle $\delta(S_g + S_{ph}) = 0$. After dropping the full derivatives with respect to the conformal time we obtain the dynamical equation for phantom cosmology from variation $\delta(S_g + S_{ph})/\delta g = 0$ as well as the dynamical equation for field from variation $\delta(S_g + S_{ph})/\delta\psi = 0$,

$$\ddot{\psi} + 3H\dot{\psi} = \frac{dU}{d\psi} + \xi R\psi. \quad (7)$$

It can be shown that for any value of $\xi$ the phantom behaves like some perfect fluid with the effective energy $\rho_\psi$ and the pressure $p_\psi$ in the form which determines the equation of state factor

$$w_\psi = \frac{-\frac{1}{2}\dot{\psi}^2 - U(\psi) - \xi[2H(\psi^2)^{\cdot} + (\psi^2)^{\cdot\cdot}] - \xi\psi^2(2\dot{H} + 3H^2)}{-\frac{1}{2}\dot{\psi}^2 + U(\psi) + 3\xi H[H\psi^2 + (\psi^2)^{\cdot}]}$$

$$\equiv \frac{p_\psi}{\rho_\psi}. \quad (8)$$

Formula (8) differs from its counterpart for the standard scalar field [52] by the presence of a negative sign in front of the term $\dot{\psi}^2$.

The second derivative $[\psi^{2\cdot\cdot}$ in the expression for the pressure in Eq. (8)] can be eliminated and then we obtain

$$p_\psi = \left( -\frac{1}{2} - 2\xi \right)\dot{\psi}^2 + \xi H(\psi^2)^{\cdot} + 2\xi(6\xi-1)\dot{H}\psi^2$$

$$+ 3\xi(8\xi-1)H^2\psi^2 - U(\psi) + 2\xi\psi\frac{dU}{d\psi}. \quad (9)$$

Of course such perfect fluid which mimics the phantom field satisfies the conservation equation

$$\dot{\rho}_\psi + 3H(\rho_\psi + p_\psi) = 0. \quad (10)$$

We can see that complexity of dynamical equation should manifest by complexity of $w_\psi$.

Let us consider the FRW quintessential dynamics with some effective energy density $\rho_\psi$ given in Eq. (8). By the quintessence we usually understand models with dark energy consisting of a dynamical cosmic scalar field. This dynamics can be reduced to the form like of a particle in a one-dimensional potential [53] and the Hamiltonian of the system is

$$\mathcal{H}(\dot{a},a) = \frac{\dot{a}^2}{2} + V(a) \equiv 0, \quad V(a) = -\rho_\psi a^4. \quad (11)$$

The trajectories of the system lie on the zero energy level for flat and vacuum models. Note that if we additionally postulate the presence of radiation matter for which $\rho_r \propto a^{-4}$ then it is equivalent to consider the Hamiltonian on the level $\mathcal{H} = E = $ const. Of course the division on kinetic and potential parts has only a conventional character and we can always translate the term containing $\dot{\psi}^2$ into a kinetic term.

Let us consider now both case of conformally and minimally coupled phantom fields.

### A. Conformally coupled phantom fields

For conformally coupled phantom fields we put $\xi = 1/6$ and rescale the field $\psi \to \phi = \psi a$. Then the energy function takes the following form for simple mechanical system with a natural Lagrangian function $\mathcal{L} = 1/2 g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - V(q)$,

$$\mathcal{E} = \frac{1}{2}\left( \dot{a}^2 + \dot{\phi}^2 \right) - \frac{\Lambda}{2}a^4 - \frac{\lambda}{2}\phi^4 - m^2\phi^2 a^2. \quad (12)$$

In contrast to the FRW model with conformally coupled scalar field the kinetic energy form is positive definite like for classical mechanical systems. The general Hamiltonian which represents the special case of two coupled nonharmonic oscillators system is

$$\mathcal{H} = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + V(q) = \frac{1}{2}(p_x^2 + p_y^2)$$

$$+ Ax^2 + By^2 + Cx^4 + Dy^4 + Ex^2y^2, \quad (13)$$

where $A, B, C, D,$ and $E$ are constants.

### B. Minimally coupled phantom fields

For minimally coupled phantom fields ($\xi = 0$) the function of energy takes the form

$$\mathcal{E} = \frac{\dot{a}^2}{2} + \frac{1}{2}(\dot{\phi}a - \phi\dot{a})^2 - \frac{\Lambda}{4}a^4 - \frac{\lambda}{4}\phi^4 - \frac{1}{2}m^2\phi^2 a^2, \quad (14)$$

where $\rho_{\text{eff}} = -1/2\dot{\psi}^2 + U(\psi), V = -\rho_{\text{eff}}a^4, \mathcal{H} = 1/2\dot{a}^2 + V(a,\psi,\dot{\psi}), \phi = a\psi, U(\psi) = 1/2m^2\psi^2 + 1/4\lambda\psi^4$ is assumed. This time we parameterize the dynamics by taking variable $\psi$ in the original cosmological time and the Lagrangian function takes the following form:

$$\mathcal{L} = \frac{a'^2}{2} + \frac{a^2\phi'^2}{2} - \frac{1}{2}m^2\phi^2 a^2 - \frac{1}{4}\lambda\phi^4 a^2 - \frac{1}{4}\Lambda a^2, \quad (15)$$

where the prime denotes the differentiation with respect to the cosmological time parameter $t$, and $V = -\rho_{\text{eff}}a^2, \rho_{\text{eff}} = -1/2\phi'^2 + U(\phi)$.

## III. NONINTEGRABILITY AS AN INVARIANT FEATURE OF PHANTOM COSMOLOGY WITH SCALAR FIELD

For a given Hamiltonian system, it is difficult to show that the system under consideration is nonintegrable. In general, there are two formulations of necessary conditions for the integrability presented by Ziglin [54,55] and Morales-Ruiz and Ramis [56,57]. Both approaches are based on a deep connection between the properties of the solutions in an enlarged complex time plane and the existence of first integrals. This idea originates from works of Kovalevskaya and Lyapunov.

Let us study the general case of the Hamiltonian system describing the conformally coupled phantom field in the FRW model of the Universe. We have

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

$$V(q_1, q_2) = \frac{1}{2}\left[\frac{\bar{\Lambda}}{2}q_1^4 + \frac{\bar{\lambda}}{2}q_2^4 - m^2 q_1^2 q_2^2\right], \quad (16)$$

where $q_1 = a, q_2 = \phi, p_1 = \dot{a}, p_2 = \dot{\phi}, \bar{\Lambda} = -\Lambda$, and $\bar{\lambda} = -\lambda$. This Hamiltonian has the natural form in which the potential is a homogeneous function of degree four with respect to both variables $a, \phi$.

From the point of view of complex dynamical behavior it is useful to distinguish from the Hamiltonian (16) with $m^2 = -\mu^2 < 0$. This case is interesting because of spontaneous symmetry breaking [58]. The Poincaré section in this case can be obtained as well as the Lyapunov exponents. In this model the chaotic behavior is present.

In other cases we can define by analogy to Castagnino *et al.* [38] the monotonic function along the trajectories. From this fact we obtain that the trajectories escape to infinity and the system has no recurrence property which guarantes the topological transitivity (the standard chaos indicators cannot be obtained).

Motter and Letelier argued that the cosmological systems with scalar fields are nonchaotic but complex in the sense of nonintegrability [42]. Moreover the nonintegrability is an invariant property of system under the coordinate change.

In the second distinguished case the complexity has the same character, and we apply the some tools to confirm the Liouville nonintegrability of this system. The Liouville integrability of the Hamiltonian system means that there is as many functionally independent functions which are in involution (Poisson brackets vanish) as is the dimension of the system.

Now we consider the problem of nonintegrability in both cases. The nonintegrability of the nonflat first case with the spontaneous symmetry breaking was investigated by Ziglin ($\Lambda = \lambda = 0$—the Yang-Mills potential) [50]. In turn, we apply the Ziglin and Morales-Ruiz and Ramis methods to flat phantom models with conformally coupled scalar fields with arbitrary parameters.

The integrability of the Hamiltonian systems with a natural Lagrangian was analyzed in details by Yoshida [59–63] in the framework of Ziglin's approach. Later Yoshida's results were sharpened by Morales-Ruiz and Ramis [56]. Note that we applied the Morales-Ruiz and Ramis result to system (16), but with the indefinite kinetic energy form $T = 1/2((p_1^2 - p_2^2))$.

The counterpart of the Hamiltonian (16) for a standard scalar field can be obtained after the canonical transformation of variables

$$q_1 \rightarrow Q_1, \quad p_1 \rightarrow P_1,$$

and

$$q_2 \rightarrow iQ_2, \quad p_2 \rightarrow P_2 = -ip_2.$$

Then of course $dp_2 \wedge dq_2 = dP_2 \wedge dQ_2$. However, in this case the phase space is complex. Moreover, trajectories have no recurrence property which guarantees the topological transitivity, which an essential element of the standard understanding of chaos.

The fundamental papers of Ziglin [54,55] gave the formulation of a very basic theorem about nonintegrability of analytic Hamiltonian systems. The Ziglin idea connects properties of solutions on a complex time plane and the existence of first integrals. This approach takes its origins in works of Kovalevskaya and Lyapunov.

The Yoshida criterion is presented in the Appendix . We apply this criterion to the analyzed system. Then the equation

$$q = V'(q), \quad q = (q_1, q_2)$$

has the following solutions:

$$z_1 = (\pm \bar{\lambda}^{-1/2}, 0),$$

$$z_2 = (0, \pm \bar{\lambda}^{-1/2}), \quad z_3 = \left(\pm\sqrt{\frac{\bar{\lambda} + \mu}{\bar{\Lambda}\bar{\lambda} - \mu^2}}, \pm\sqrt{\frac{\bar{\lambda} + \mu}{\bar{\Lambda}\bar{\lambda} - \mu^2}}\right).$$

The integrability indices for this points are

$$\lambda_i = -\operatorname{tr} V''(z_i) - 3, \quad i = 1, 2, 3$$

and

$$\lambda_1 = \frac{\mu}{\Lambda}, \quad \lambda_2 = \frac{\mu}{\lambda}, \quad \lambda_3 = \frac{\lambda_1 \lambda_2 - 2(\lambda_1 + \lambda_2) + 3}{1 - \lambda_1 \lambda_2}, \quad \mu = m^2. \quad (17)$$

Thus, from the Yoshida criterion it follows that if there exists $l \in \{1, 2, 3\}$ such that $\lambda_l \in N_4$ then system (16) has no additional meromorphic first integral that is functionally independent of $H$. Moreover, our previous application of the Morales-Ruiz and Ramis result to the considered system gives that if we introduce quantities (17) and three discrete sets

$$I_1 = \{p(2p - 1)|p \in \mathbb{Z}\},$$

$$I_2 = \{1/8[-1 + 16(1/3 + p)^2]|p \in \mathbb{Z}\},$$

$$I_3 = \{1/2[3/4 + 4p(p - 1)]|p \in \mathbb{Z}\}, \quad (18)$$

then if $\lambda_1, \lambda_2, \lambda_3 \notin I = I_1 \cup I_2 \cup I_3$ the system is nonintegrable. Therefore, only for certain values of model parameters the phantom cosmology is integrable. We can conclude that the Liouville nonintegrability is the generic property of the system.

If we consider a nonflat model then the effects of curvature are negligible near the singularity and the considered case describes a generic situation. In this way the phantoms give rise to the complex dynamics in the sense of nonexistence of a sufficient number of independent first integrals. As a consequence, we can express some scepticism about prediction for the equation of state factor $w(z)$ in the presence of the phantom component of the dark energy.

Our conclusion is that in a generic case the phantom scalar field can produce the complex behavior. The complexity of dynamics is formulated in terms of nonintegrability (i.e., nonexistence of an additional first integral) because a standard understanding of chaos has no significant physical meaning in the context of a gauge freedom in the choice of a lapse function (time parametrization).

Beck [64] proposed an interesting idea that stochastically quantized scalar fields can offer some solution to the cosmological coincidence problem of $\Lambda$. In this approach the chaotic fields have a classical equation of state close to $p = -\rho$, i.e., that the chaotic fields naturally generate a small cosmological constant. It is possible that phantoms are just a phenomenological description of this situation on a purely classical level.

Let us also note that the regular behavior of dynamics in phantom cosmology can appear to be different from the considered types of potentials [65] but $V(\phi) \propto \phi^2$ is the simplest one in which this phenomenon occurs. Moreover it can only appear if we treat phantom energy in terms of a single scalar field.

## IV. PHANTOM COSMOLOGY IN TERMS OF THE BAROTROPIC EQUATION OF STATE VIOLATING THE WEAK ENERGY CONDITION

It is well known that for a given evolution of the model it is possible to construct a potential for a minimally coupled scalar field which would reproduce this cosmological evolution [66]. Sometimes it is possible to find that the explicit form of a scalar field potential can reproduce the evolution arising in some perfect fluid cosmological model [67].

The very different picture is found, if we consider phantom energy as a some kind of perfect fluid with supernegative pressure then, in the contrast to the previous case, the dynamics is regular at thevery beginning.

Let us consider the dynamics of the FRW models with phantoms where the specific form of the equation of state for phantom fluid is assumed. We model the fluid which violates the weak energy condition using the equation of state $p = w\rho$ and $w = \text{const} < -1$. Such a model of fluid can be treated as the simplest phenomenological model of phantom matter.

The dynamics of this model can be represented by a two-dimensional dynamical system (therefore nonchaotic at very beginning) on the phase plane $(x, \dot{x}) \equiv (x, y)$ or by motion of a classical particle in the one-dimensional potential $V(x): x = a/a_0$ [53], i.e.,

$$\dot{x} = y,$$

$$\dot{y} = -\frac{\partial V}{\partial x}. \qquad (19)$$

System (19) has the first integral of energy in the form

$$\frac{\dot{x}^2}{2} + V(x) = 0, \qquad (20)$$

where

$$V(x) = -\frac{1}{2}(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\mathrm{ph},0}x^{-3(1+w)+2} + \Omega_{k,0}). \qquad (21)$$

For the mixture of noninteracting matter and phantoms here $\Omega_{i,0}$ is the density parameters at the present epoch. In the general case the potential of the particle-universe takes the form

$$V(x) = -\frac{1}{2}\sum_i \Omega_{i,0}x^{-3(1+w_i)+2}, \qquad (22)$$

where $w_i = -1$ for the cosmological constant, $w_i = -1/3$ for string fluid (also curvature fluid), and $w_i = -2/3$ for topological defects.

The phase portraits for the model described by system (19) for the potential function (21) are shown on Fig. 1. The trajectory of the flat model separates the regions of closed ($\Omega_{k,0} < 0$) and open ($\Omega_{k,0} > 0$) models. Moreover, both phase portraits are topologically equivalent. The presence of additional terms like strings, topological defects (see Ref. [31]) do not change the structure of the phase plane. There is the single critical point located on the $x$ axis as an intersection with the boundary of the strong energy condition $\rho + 3p \geqslant 0$.

Note that the obtained phase portraits are equivalent to phase portraits of the FRW model with the cosmological constant.

From the first integral (20) for the mixture of the cosmological constant and phantom type matter in the flat FRW models we obtain the relation

$$(\ln x)^2 = \left(\frac{\dot{x}}{x}\right)^2 = \Omega_{\mathrm{ph},0}x^{-3(1+w)} + \Omega_{\Lambda,0}, \qquad (23)$$

which preserves its form structure under the change both a position variable and a sign of the quintessential parameter $(w+1)$,

$$x \rightarrow \frac{1}{x}, \quad (1+w) \rightarrow -(1+w). \qquad (24)$$

Therefore, if a phantom epoch exists its dynamics can replicate the corresponding evolution for the subnegative equation of state (for example $w = -4/3$ corresponds to $w = -2/3$). From this kind of symmetry we obtain that if $x(t)$ is the solution of (23) for the subnegative equation of state $p = w\rho$
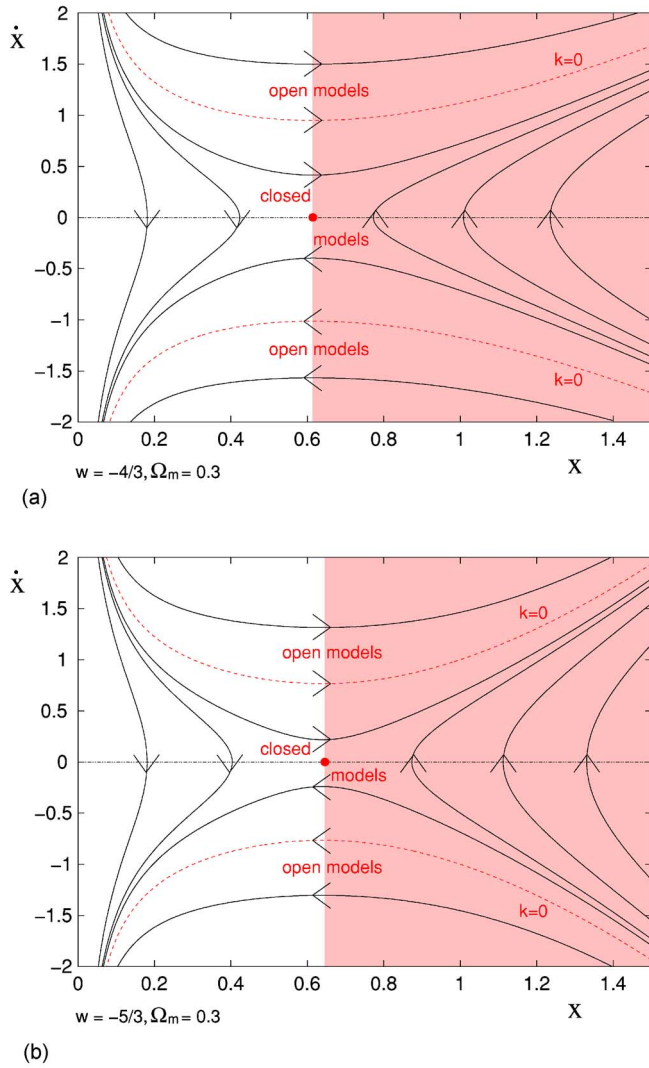
(a)



(b)

FIG. 1. (Color online) The phase portraits for the FRW models with phantom matter described by the equation of state $p=w\rho$ for (a) $w=-4/3$ and (b) $w=-5/3$. The dashed lines are the flat model trajectories. The shaded region is the region of accelerated expansion of the Universe. Note the topological equivalence of both phase portraits.

then $x^{-1}(t)$ is also its solution for other form of the negative equation of state $p=-(w+2)\rho$. Let us note that for $w=-1$ the duality relation which is motivated by superstring theory of duality symmetries [68,69] is the exact symmetry of dynamical equations.

## V. CONCLUSIONS

This paper addressed the problem of complexity of the flat FRW dynamics with phantom modeled in terms of scalar fields. We proposed a criterion of nonintegrability in the Liouville sense as an adequate measure of complexity of the phantom cosmology. This approach is opted because the gauge freedom in the choice of a time parameterization or a lapse function unables us to discuss chaos in general relativity in the standard way. Nonintegrability is an invariant fea-

ture of a system and can be use as an indicator of its complex dynamics.

We considered the two approaches to model the phantom fields in the FRW model and showed how different dynamics of the models are in these approaches. In the first approach which could be called "microscopic" we used scalar fields in modeling the pressure and energy density. In the second approach we used the explicit dependence of pressure versus energy density, which can be called phenomenological and found integrable dynamics in contrast to the nonintegrable dynamics of the first approach.

In this context nonintegrability is a generic feature of the phantom cosmology and only for a certain discrete set of values of model parameters phantom cosmology is integrable in the Liouville sense. From the physical point of view one can interpret this property as the complexity of dynamical behavior of trajectories in the phase space. Therefore we cannot expect any simple analytical relation for solutions (trajectories) of the system or the form of relation $p(\dot{\phi},\dot{a},\phi,a)$ and $\rho(\dot{\phi},\dot{a},\phi,a)$ along the trajectories.

Nevertheless, the integrable cases with zero measure in the space of all solutions exists, they can lead to some analytical dependence of $p(\rho)$, after the elimination of time. As the example of simple analytical form of the equation of state we considered the barotropic equation of state which violates the weak energy condition. We obtained that this model is integrable and exhibits the regular dynamics.

We conclude that nonintegrability which is a generic feature of the FRW model with phantom fields favors rather nonanalytical forms of the equation of state. However, assuming the barotropic form of the equation of state for the phantom model we obtain the integrable dynamics at very beginning. We expect that phenomenological equation of state can be realized by a microscopic scalar field with some potential. Therefore, we assume at very beginning the existence of some relation (a first integral) between scalar fields, its derivative and evolutional parameter of the Universe. From the mathematical point of view this requirement means the existence of some invariant in the phase space. If we prove nonintegrability, than there no such relation in the considered class of potentials. If there is even a discrete set of parameters for which the system is integrable we have hope for finding this relation.

## APPENDIX: THE OUTLINE OF NONINTEGRABILITY CRITERION

Here we present only the facts needed for a formulation of a criterion in possibly simple settings. We consider a complex symplectic manifold $\mathbb{C}^{2n}$ with the canonical symplectic structure $\Omega$. A Hamiltonian vector field $v_H$ is determined by a complex Hamiltonian function $H:\mathbb{C}^{2n}\to\mathbb{C}$ by the relation $\Omega(v_H,\cdot)=dH$. We assume that Hamilton's analytic equations

$$\frac{dz}{dt} = v_H(z), \quad z = (z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n}, \quad t \in \mathbb{C}, \quad \text{(A1)}$$

have the nonequilibrium solution $z = \varphi(t)$. To simplify the exposition we assume that this solution lies on a two-dimensional invariant plane

$$\Pi = \{(z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n} | z_i = 0, \quad i = 1, \ldots, 2(n-1)\}.$$

The phase curve $\Gamma = \{\varphi(t) \in \mathbb{C}^{2n} | t \in \mathbb{C}\}$ is a Riemannian surface with a local coordinate $t$. Together with equations (A1) we consider also variational equations along solution $\varphi(t)$,

$$\frac{d\xi}{dt} = A(t)\xi, \quad A(t) = \frac{\partial v_H}{\partial z}[\varphi(t)]. \quad \text{(A2)}$$

This system separates into the normal and the tangential subsystems. In our settings this separation takes a very simple form—the matrix $A(t)$ has a block diagonal structure. We consider the normal variational equations (NVE),

$$\frac{d\eta}{dt} = B(t)\eta, \quad \eta \in \mathbb{C}^{2(n-1)}, \quad \text{(A3)}$$

where $B(t)$ is $2(n-1) \times 2(n-1)$ upper diagonal block of matrix $A(t)$. We choose a point $t_0 \in \mathbb{C}$ and a matrix of fundamental solutions of the NVE $X(t)$, defined in a neighborhood of $t_0$. With a close path $\alpha$ on complex time plane starting and ending at point $t_0$ we can associate a matrix $S \in \mathrm{GL}(2(n-1), \mathbb{C})$ in the following way. We integrate NVE (A3) along the path $\alpha$, i.e., we make an analytic continuation of $X(t)$ along this path. As a result from the fundamental solution $X(t)$ we obtain another fundamental solution $Y(t)$. From the general theory of linear systems it follows that $Y(t) = SX(t)$ for some $S \in \mathrm{GL}(2(n-1), \mathbb{C})$. Because the system is Hamiltonian, $S$ is a symplectic matrix, i.e, $S \in \mathrm{Sp}(2(n-1), \mathbb{C})$. In this way, considering all possible paths, we obtain a matrix representation of the first homotopy group $\pi_1(\Gamma)$ of $\Gamma$. It forms a finitely generated subgroup of $\mathrm{Sp}(2(n-1), \mathbb{C})$ and it is called a monodromy group. We denote it $M$.

Let us take an element of the monodromy group $g \in M$. Its spectrum has the form

$$\mathrm{spectr}(g) = (\lambda_1, \lambda_1^{-1}, \ldots, \lambda_{n-1}, \lambda_{n-1}^{-1}), \quad \lambda_i \in \mathbb{C}.$$

The element $g$ is resonant if

$$\prod_{l=1}^{n-1} \lambda_l^{k_l} = 1 \text{ for some} (k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1} \setminus \{0\}.$$

**Theorem 1 (Ziglin [54])** *Let us assume that there exists a nonresonant element $g \in M$. If the Hamiltonian system possesses in a connected neighborhood of $\Gamma$ $n-1$ meromorphic first integrals which are functionally independent with $H$ then for an element $g' \in M$: if $ge = \lambda e$ for $\lambda \in \mathbb{C}$ and $e \in \mathbb{C}^{2(n-1)}$, then $g(g'e) = \lambda'(g'e)$ for some $\lambda' \in \mathbb{C}$.*

In the case of a system with two degrees of freedom this theorem can be formulated in a more operational way.

**Theorem 2** *Let us assume that there exists a non-resonant element $g \in M$. If there exists other element $g' \in M$ such that*
  (1) *$\mathrm{tr}\, g' \neq 0$ and $gg' \neq g'g$, or*
  (2) *$\mathrm{tr}\, g' = 0$ and $gg'g \neq g'$,*

*then there is no additional meromorphic first integral functionally independent of $H$ in a connected neighborhood of $\Gamma$.*

The main difficulty with the application of the Ziglin theorem is the determination of the monodromy group of the NVE. Only in very special cases we can do this analytically. Yoshida [59–62] developed the Ziglin approach for these cases when the Hamiltonian of a system has the natural form and the potential is a homogeneous function. In this case a particular solution can be found in the form of "straight line solution" and the NVEs for it can be transformed to a product of certain copies of hypergeometric equations for which the monodromy group is known. This allows one to formulate adequate theorems in a form of an algorithm. Below we describe it for the Hamiltonian system with two degrees of freedom.

Consider the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \quad (q_1, q_2, p_1, p_2) \in \mathbb{C}^4, \quad \text{(A4)}$$

where $V(q_1, q_2)$ is the homogeneous function of degree $k$, i.e.,

$$V(Cq_1, Cq_2) = C^k V(q_1, q_2). \quad \text{(A5)}$$

In a generic case this system has a straight line solutions of the form

$$q_1 = C_1 \phi(t), \quad q_2 = C_2 \phi(t) \quad \text{(A6)}$$

where $\phi(t)$ is a solution of a nonlinear equation

$$\ddot{\phi} = -\phi^{k-1}$$

and $(C_1, C_2) \neq (0,0)$ are solutions of the following system:

$$C_1 = \partial_1 V(C_1, C_2), \quad C_2 = \partial_2 V(C_1, C_2). \quad \text{(A7)}$$

The variational equations take the form

$$\begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \end{bmatrix} = - \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} [\phi(t)]^{k-2},$$

where $V_{ij} = \partial_i \partial_j V(C_1, C_2)$ for $i, j = 1, 2$. Since the Hessian of $V$ is symmetric it is diagonalizable by an orthogonal transformation and the system separates to

$$\ddot{\xi} = -\lambda_1 \Phi^{k-2}(t)\xi, \quad \text{(A8)}$$

$$\ddot{\eta} = -\lambda_2 \Phi^{k-2}(t)\eta, \quad \text{(A9)}$$

where $\lambda_1, \lambda_2$ are real eigenvalues of the Hessian. Let us note that it is not true for indefinite systems where the Hessian is not a symmetric matrix.

It can be shown that the Hessian of $V$ at $C = (C_1, C_2)$ has the eigenvalue $\lambda_1 = k - 1$. Thus, its second eigenvalue is equal $\lambda_2 = \mathrm{tr}\, V(C_1, C_2) - (k-1)$, and it is called the integrability index. Equation (A9) can be transformed to the hypergeometric equation. The monodromy matrices of this equation are parameterized by $\lambda$ and conditions of the Ziglin theorem put a restriction on values of $\lambda$—simply, we can identify those

values of $\lambda$ for which the system is not integrable (more precisely: it does not possess an additional meromorphic first integral). To state it accurately let us define

$$I_k(p) = \left( \frac{k}{2} p(p+1) - p, \frac{k}{2} p(p+1) + p + 1 \right), \quad p \in \mathbb{N}, \quad (A10)$$

and

$$N_k = \mathbb{R} \setminus \bigcup_{p \in \mathbb{N}} I_k(p). \tag{A11}$$

Then it follows that Hamiltonian system (A4) with homogneous potential (A5) of degree $k$ is not integrable if the integrability index $\lambda$ corresponding to a certain straight line solution (A6) belongs to $N_k$. Let us note that equations (A7) usually have several solutions and thus it is necessary to check the Yoshida criterion for each of them.

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